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Limit Cycle Predictions of a Nonlinear Journal-Bearing System

An analysis is presented of the self-excited vibrations of a journal carried in a cylindrical fluid film bearing. Using linear stability theory, the values of the system parameters at the point of loss of stability are determined. These values agree well with those of previous investigators. Solutions of the nonlinear system equations are obtained by time discretization and by an arc-continuation method for solving the obtained nonlinear algebraic equations. In this way periodic solutions of the nonlinear equations of motion are calculated as a function of the system parameters. The behavior of the journal can be explained by the results of these calculations.

1 Introduction

Rotors carried in fluid-film journal bearings may develop, under certain operating conditions, an instability known as oil-whip or oil-whirl. This instability manifests itself by a growth of the whirl amplitude. This amplitude may become so large that it may endanger the safe operation of the bearing or damage it [1, 2]. The frequency of the oscillation is about half the rotational frequency of the rotor. Rotational energy is transformed by the fluid forces from the rotor, which rotates with constant speed, into the whirl to sustain the oscillation. It is a purely self-excited vibration and it does not result from imbalance.

By linearization of the equations of motion about the equilibrium position Lund and Saibel [3] and Hollis and Taylor [4] determined on which combination of important parameters the rotor becomes unstable and starts whirling. Applying the method of averaging, Lund and Saibel also examined the existence of periodic motions in excess of the threshold speed. Hollis and Taylor [4] did the same, using the Hopf bifurcation theory. Their examinations were limited to the analysis of small periodic motions in the neighborhood of the equilibrium position on the linear stability threshold. In this paper the behavior of a simple rotor-bearing system will be examined in some detail. Firstly, the values of the dimensionless system parameters for which the journal loses its stability are calculated. This parameter set corresponds to a linear stability threshold which is in agreement with the results presented by other authors [3, 4]. Secondly, a method will be discussed which provides periodic solutions of the nonlinear equations of motion as a function of the system parameters. It will be shown that periodic motions may even exist before the threshold of linear stability is reached. The results enable us to explain the behavior of the rotor-bearing system near this stability threshold.

2 System Equations

In this section the governing equations of motion of a rotating journal mounted in a cylindrical fluid film bearing will be derived and transformed into dimensionless form. The resulting equations contain three dimensionless parameters which entirely characterize the system.

Consider a journal carried in a single fluid bearing, as illustrated in Fig. 1. The applied load F_c on the journal is constant both in direction and magnitude. Let the rotor mass be M and the journal center displacements x and y. C is the radial bearing clearance, Ω the angular velocity, B the bearing length, D the bearing diameter, η the lubricant viscosity, and t the time. The equations of motion are:

$$M \frac{d^2x}{dt^2} = F_x\left(x, y, \frac{dx}{dt}, \frac{dy}{dt}, \eta, B, D, \Omega, C\right) \qquad (1)$$

$$M \frac{d^2y}{dt^2} = F_y\left(x, y, \frac{dx}{dt}, \frac{dy}{dt}, \eta, B, D, \Omega, C\right) - F_c \qquad (2)$$

In equations (1) and (2) F_x and F_y are the hydrodynamic forces exerted by the lubricant film on the journal in the x and y direction, respectively. By solving the Reynolds equation over the bearing with appropriate boundary and cavitation conditions, the pressure distributions around the bearing can be
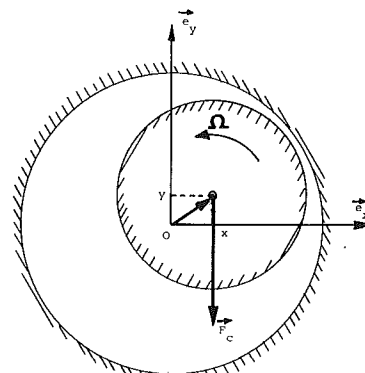

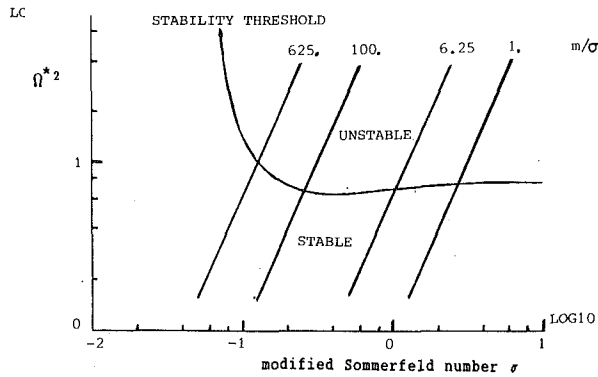
Fig. 1 Journal bearing configuration

Fig. 2 The linear stability threshold in the σ-Ω^{*2} plane for the short bearing model with $\beta = 0.25$

found. Integration of the pressure yields the bearing force on the journal. Here an approximate analytical solution is used, known as the Ockvirk short bearing solution [3, 4]. This solution provides force values which are in close agreement with actual results for length-diameter ratios up to 0.5. Here, we will restrict ourselves to the case B/D equals 0.25.

Equations (1) and (2) are written in a dimensionless form using C as the characteristic length and using the average rotational velocity of rotor and its housing ω_c ($= 0.5 \, \Omega$) as the characteristic frequency.

$$x^* = x/C \qquad\qquad y^* = y/C$$

$$\tau = \omega_c t \qquad\qquad \sigma = \frac{1}{4} \frac{\eta \omega_c D B}{F_c}$$

$$\Omega^* = \Omega \sqrt{\left(\frac{CM}{F_c}\right)} \qquad m = \Omega^{*2}/\sigma$$

$$F_x^* = \frac{1}{\sigma} \frac{F_x}{F_c} \qquad\qquad F_y^* = \frac{1}{\sigma} \frac{F_y}{F_c}$$

$$\beta = B/D \qquad\qquad \dot{x}^* = \frac{dx^*}{d\tau}, \text{ etc.} \qquad (3)$$

Parameter σ is called the modified Sommerfeld number and m the dimensionless journal mass. The equations of motion (1) and (2) become:

$$m\ddot{x}^* = F_x^* \, (x^*, y^*, \dot{x}^*, \dot{y}^*, \beta) \qquad (4)$$

$$m\ddot{y}^* = F_y^* (x^*, y^*, \dot{x}^*, \dot{y}^*, \beta) - 1/\sigma \qquad (5)$$

These two nondimensional equations are characterized only by m, β, and σ.

3 Equilibrium Positions and Their Stability

In this section, we will discuss the results of a linear stability analysis that yields combinations of the system parameters Ω^{*2} and σ at which the rotor becomes unstable.

As β has been set equal to 0.25, the time independent equilibrium positions of the journal only depend on the value of the modified Sommerfeld number. The linear stability of the equilibrium points can be investigated after linearization of the equations of motion about these points. The linear stability analysis results in a stability threshold in the $\sigma - \Omega^{*2}$ plane. In Fig. 2, this stability threshold in the $\sigma - \Omega^{*2}$ plane is plotted for $\beta = 0.25$, using logarithmic axes. For small values of σ and Ω^{*2} the rotor is stable and for large values it is unstable. The results shown in Fig. 2 completely agree with the results obtained by Hollis and Taylor [4]. The variable σ along the horizontal axis is linearly dependent on the rotor speed; while the variable Ω^{*2} along the vertical axis depends quadratically on the rotor speed. In Fig. 2 lines are drawn that represent a

set of system states where only the rotor speed varies and all other variables are constant. Hollis and Taylor [4] called such lines "constant bearing lines." The nondimensionalization (3) introduces the rotor speed in both σ and m and therefore, each line can be indicated with the ratio between m and σ. Using logarithmic axes, the constant bearing lines are straight lines in the $\sigma - \Omega^{*2}$ plane as drawn in Fig. 2.

It can be seen from Fig. 2 that the stability threshold has a minimum value if Ω^{*2} is equal to approximately 6.4 while σ is equal to about 0.4. Both for higher and lower values of σ, the stability threshold increases in terms of Ω^{*2}. For higher values of σ, the stability boundary reaches a maximum for Ω^{*2} somewhere near the value 7.4. For low values of σ and Ω^{*2}, no limit seems to be present for the values of Ω_t^{*2} and σ_t, respectively, at the stability threshold.

Mathematically, linear stability is investigated by solving the corresponding classical eigenvalue problem of the linearized homogeneous equations of motion. At the stability threshold, the eigenvalue problem yields two purely conjugate imaginary eigenvalues. It can be verified (and this is actually done by Hollis and Taylor [4]) that a Hopf bifurcation occurs at the linear stability threshold. According to the Hopf bifurcation theory, periodic orbits exist which bifurcate from the threshold of linear stability [5]. In order to predict these periodic orbits regardless of their stability and taking the complete nonlinear characteristics of the bearing into account, a suitable solution method is now introduced.

4 Solution Method

A method is discussed which provides periodic solutions of the nonlinear equations of motion (4) and (5) as a function of any combination of the system parameters. This method is based on time discretization of the system equations combined with a numerical solution algorithm to solve the resulting nonlinear algebraic equations.

Instead of a continuous periodic solution of (4) and (5) an approximate solution at a discrete number of times will be determined. In order to discretize the equations (4) and (5) let τ_j ($j = 1, \ldots, N$) be an equidistant partition of time T, yielding:

$$\tau_j = \frac{(j-1)}{N} T \qquad (6)$$

with T the unknown period of time of the periodic solution. The velocity and acceleration both in x^*-direction and y^*-direction at τ_j are expressed with a 4th order central difference scheme. Application of this discretization scheme in the equations of motion at N discretization points yields $2N$ algebraic equations. Without loss of generality it is assumed that at $\tau = 0$ the velocity in the direction of x^* is equal to zero. The resulting $2N + 1$ algebraic equations are denoted by:

$$G(z) = 0 \qquad (7)$$

with the $2N + 1$ unknowns:

$$z^* = [x^*(\tau_1), \ldots, x^*(\tau_N), y^*(\tau_1), \ldots, y^*(\tau_N), T] \qquad (8)$$

This set of algebraic equations can be solved straight-forward by a standard multidimensional Newton-Raphson method. If one of the design variables or a fixed combination of design variables is added the equations can be solved by an arc-continuation method [6]. In this way, solutions are obtained as a function of that variable or that combination of variables.

The operation of the arc-continuation method is based on a prediction step followed by correction steps until the solution at a next value of the design variable is reached.

In order to find an initial solution for starting the Newton Raphson procedure, the Hopf bifurcation theory is used [5]. Following this theory, equilibrium point data, eigenvalue data
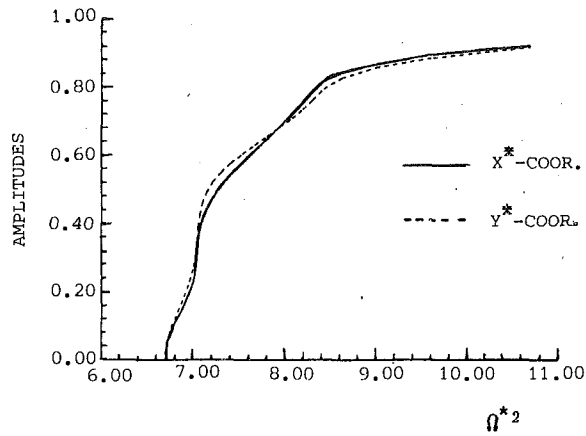
Fig. 3 Amplitudes of periodic solutions that branch off via a Hopf bifurcation at the linear stability threshold for the short bearing model, m/σ kept equal to 100
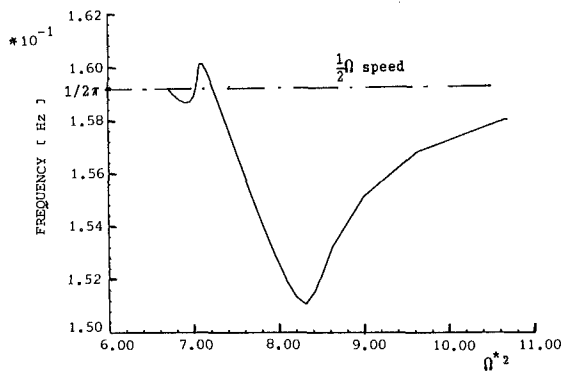

Fig. 4 Frequencies of the periodic solutions with respect to the dimensionless time vs. Ω^{*2} for the results shown in Fig. 3

Fig. 5 Some periodic solutions in the x^*-y^* plane at different values of Ω^{*2} from the results shown in Fig. 3

Fig. 6 Amplitudes of the x^*-coördinate of existing periodic solutions in the σ-Ω^{*2} plane for the short bearing model

and eigenform data provide an approximation of small orbits near the equilibrium point at the stability threshold. Detailed information on numerical aspects, comparison with other techniques, error estimation, etc. is given in [7] and [8].

5 Periodic Solutions

In this section, some results concerning the periodic solutions are presented and discussed. These essentially nonlinear results illustrate the influence of nonlinearities of the bearing model on the behavior of the rotor near the stability threshold.

As an example, the results of a series of periodic solutions are shown as a function of the design variables. The data corresponding to one of the points at the stability threshold shown in Fig. 2 is:

equilibrium point: $x_t^* = 0.4211$ $y_t^* = -0.3534$

eigenvalue: $\omega_t = 0.0 + 1.0003i$

eigenform: $u_{tx} = 0.6657$ $u_{ty} = 0.5066 + 0.5478i$

design variables: $\sigma_t = 0.2591$ $\Omega_t^{*2} = 6.7151$

With this data, it is possible to estimate the form and frequency of periodic solutions that branch off. Rather than calculating periodic solutions as a function of the existing design variables, the solutions for a fixed m/σ ratio ($m/\sigma = 100$) are calculated, thus as a function of rotor speed.

In Figs. 3 and 4, the results of this calculation are presented in order to show the solutions as a function of Ω^{*2}. Since we are dealing with a constant bearing line, we must be aware that a change of Ω^{*2} will be proportional to a change of σ.

The amplitudes of the periodic solution in both the x^*- and y^*-directions have been calculated as the maximum value minus the minimum value divided by two in each of these directions.
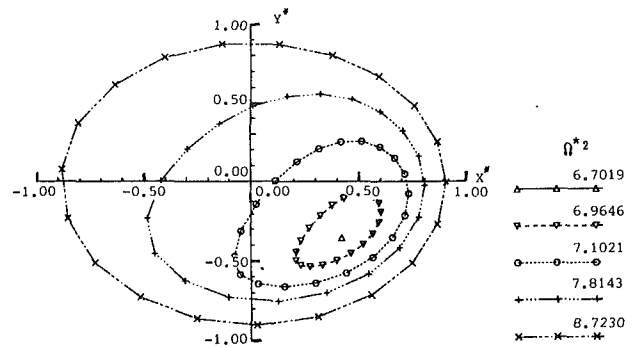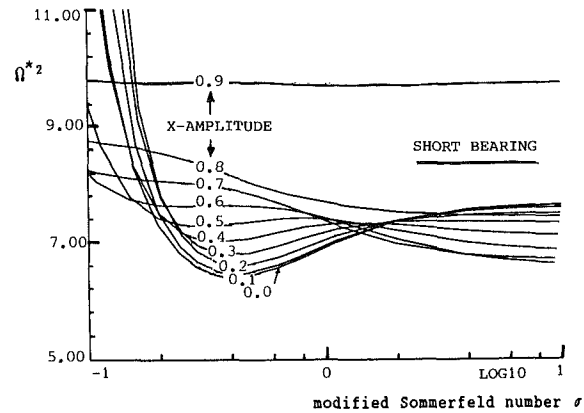
The number of discretization points that is used was equal to 40.

Figure 5 depicts the periodic solutions in the x^*-y^* plane, corresponding to some of the points shown in Fig. 3. Solution methods like the harmonic balance method used by Lund and Saibel [3] and Hollis and Taylor [4] will only predict elliptical solutions. Figure 5 shows that this assumption was not acceptable even for periodic solutions with very small amplitudes.

For a constant bearing line, an increase of Ω^{*2} agrees with an increase in the rotor speed. Figure 3 shows that, for an increase of the rotor speed, the amplitudes of a periodic solution will increase.

The unstable equilibrium position of the shaft makes it likely that the shaft will start to whirl with the predicted periodic solution. When such a periodic solution has a small amplitude, the rotor behavior can be quite acceptable; however, it can be seen that the periodic solutions grow rapidly even for a small increase of Ω^{*2}. In practical situations, therefore, it will be very difficult to recognize such motions.

In Fig. 4, the radial frequency of the periodic solution is given with respect to the dimensionless time. All solutions have a radial frequency that equals approximately half the radial rotor frequency, as noticed in practice.

If calculations are carried out as shown in Fig. 3 for a number of different points at the stability threshold, an impression of the rotor behavior will be obtained near the stability boundary. In Fig. 6, the results of these calculations are shown. In the σ-Ω^{*2} plane, lines are drawn for which the periodic solutions have the same amplitude in the x^*-direction. In order to get a clearer view near the stability threshold, only a part, but an important part, of the plane has been shown. The curve indicated by x^*-amplitude = 0.0 matches with the curve shown in Fig. 2. But the values of Ω^{*2} are in this case not on a

logarithmic scale and, therefore, it is difficult to draw the constant bearing lines. One of the things that can be observed, although it can be seen more clearly in a picture that shows a larger part of the σ-Ω^{*2} plane, is that the existence of periodic solutions with large amplitudes (>0.7) does not depend on the value of σ. Particularly for low values of σ, it means that periodic solutions exist with large amplitude although the linear stability threshold is not reached. Here, one has to be aware of a restricted area for which the static equilibrium solution is attracting.

Although the values of the amplitudes grow rapidly over a small range of values for Ω^{*2}, it is interesting to see how they grow as a function of increasing rotor speed. For high values of σ, periodic solutions with small amplitudes can be found at rotor speeds that are less than the rotor speeds at the onset of instability. At an amplitude of approximately 0.7, the slowest rotor speeds occur at which a periodic solution still can be found. For values of σ between 0.125 and approximately 1.0, an increased rotor speed above the stability threshold will mean that the amplitude of the existing periodic solution is slowly increasing. When the stability threshold in that area is reached, it may be possible to measure the whirl orbit of the rotor. With a 10 percent increase in Ω^{*2} value above the stability threshold, the amplitudes are sometimes still smaller than 0.5. For values of σ below 0.12, periodic solutions with a small amplitude exist above the linear stability threshold. Although Fig. 6 shows only a narrow band of the periodic solutions with low amplitudes, the increase of the rotor speed along those constant bearing lines where these solutions still exist can be considerable. For higher amplitudes in this area of σ-values, periodic solutions are found for much lower values of Ω^* than the values for Ω^*_t at the stability threshold.

6 Conclusions

The behavior of a two-degrees-of-freedom model of a rotor-bearing structure has been reported. The quasistatic equilib-rium positions of the unexcited rotor lose their stability when the rotational speed of the rotor increases. The onset of instability is called the linear stability threshold. The existence of self-excited periodic motions of the rotor center near the linear stability threshold follows from the Hopf bifurcation theory. Both for high and for low values of the modified Sommerfeld number, it is to be expected from the results that the rotor suddenly jumps to a large orbit when the linear stability threshold is exceeded. For moderate values a more gradual transition is to be expected. It becomes clear from interpreting Fig. 6 that there are some values for the modified Sommerfeld number where experimental results will be extremely sensitive to small variations of the design variables and, therefore, practically hard to verify.

For large amplitudes of the predicted motions, the periodic motion does not depend on the value for the modified Sommerfeld number.

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