Packing fraction of geometric random packings of discretely sized particles

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The packing fraction of geometric random packings of discretely sized particles is addressed in the present paper. In an earlier paper [Brouwers, Phys. Rev. E **74**, 031309 (2006);**74**, 069901(E) (2006)], analytical solutions were presented for the packing fraction of polydisperse geometric packings for discretely sized particles with infinitely large size ratio and the packing of continuously sized particles. Here the packing of discretely sized particles with finite size ratio u is analyzed and compared with empirical data concerning five ternary geometric random close packings of spheres with a size ratio of 2, yielding good agreement.

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In Ref. [1] analytical solutions were presented for the packing fraction of polydisperse geometric packings. Two cases were distinguished, packings of discretely sized particles with infinitely large size ratio u and the packing of continuously sized particles. It was shown that the latter distribution follows from a discretely sized packing by taking the limit $u \to 1$.

The first case, $u \to \infty$, concerns noninteracting size groups, and the void fraction of the assembly can be derived from the packing fraction of the monosized packing of the particle shape considered (φ_1) [2]. For a bimodal packing fraction, the minimum void fraction thus becomes φ_1^2, in Ref. [3] designated as a saturated packing. In Fig. 1 this principle is illustrated by the void fraction of packed bimodal broken particles with $\varphi_1 = 0.5$, taken from [3]. The figure reveals that the bimodal void fraction h is governed by the size ratio u and the composition of the mix. The composition can be expressed in the volume fraction of the large component (c_L) or the volume ratio of large and small components [$r = c_L/c_S = c_L/(1–c_L)$], so $h(u, r)$. For infinitely large size ratio, one can see that the minimum bimodal packing fraction tends to 0.25, corresponding to φ_1^2. The composition then corresponds to $c_L = 2/3$ ($r = 2$), corresponding to $1/(1 + \varphi_1)$ [1].

Extending this situation of two noninteracting particle factions to n fractions yields φ_1^n as the saturated void fraction. For such saturated packing, i.e., a packing where the concentrations are such that a size class can fill the voids of the next larger size class, both the size ratio and the concentration ratio of subsequent particle classes is constant [1]. From the aforesaid void fraction expression one can see that the void fraction is reduced proportionally to the number of size groups minus 1, and is homogeneous on some long length scale. The bimodal void fraction, described by the function $h(u, r)$, is also defined in the vicinity of $u = 1$. Considering that bimodal void fraction packing $h(u, r)$ can range between φ_1 and φ_1^2, in Ref. [1] it was derived that the monosized void fraction φ_1 is reduced with a factor h/φ_1 when a second smaller fraction is added. When the size ratio u between the adjacent sizes in a multicomponent packing is finite, the perfect packing of smaller particles in the voids of the larger ones does not hold anymore, but also in this case the void fraction reduction involved with the size ratio of adjacent size groups (of constant ratio) is proportional to the number of size groups. Accordingly in Ref. [1], for a system with n size groups, was proposed

$$\varphi = \varphi_1 \left[\frac{h(u,r)}{\varphi_1} \right]^{n-1}. \tag{1}$$

A maximum packing fraction is obtained when h is minimal. For a saturated packing of n sizes, Eq. (1) readily yields φ_1^n indeed. The effect of adding an infinite number of size groups, to obtain a continuous packing, on the void fraction was also examined in Ref. [1]. Adding more size groups to the mix will reduce the void fraction. But on the other hand, its effect is less as the size ratio of adjacent groups tends to unity (i.e., $u \to 1$) and the resulting void fraction of adjacent size groups, governed by $h(u, r)$, tends to φ_1. This limit was solved and the void fraction of a geometric continuous packing obtained. This expression was extensively validated by comparing it with a broad set of empirical data of [4], concerning continuous geometric packings of broken particles.

That Eq. (1) also holds for a finite number of interacting particles with finite size ratio is verified here. This is done by considering empirical ternary sphere packing results of Jeschar *et al.* [5], who did not specify how their packings were prepared and measured. They obtained $\varphi_1 = 0.366$ for the monosized packing of spheres with diameters 7 mm (d_3), 14 mm (d_2), and 28 mm (d_1). These values are compatible with random close packing (RCP) values from other experiments [6] and with computer generated values [7]. Equation (1) is, however, applicable to any mode of packing, from random loose to random close packed. To apply the current multimodal packing model, unimodal, bimodal, and trimodal packings only need to be prepared and compacted in a comparable way [1].

The bimodal and trimodal packing results from [5] are summarized in a ternary plot (Fig. 2). The sides of the triangle show that the binary packings of 7 and 14 mm have the same void fraction as the mixes of 14 and 28 mm packing at equal compositions, which would be expected as the size ratios of these binary mixtures are equal ($d_1/d_2 = d_2/d_3 = u = 2$). These measured binary void fractions h are included in Table I for various compositions (c_L or r).

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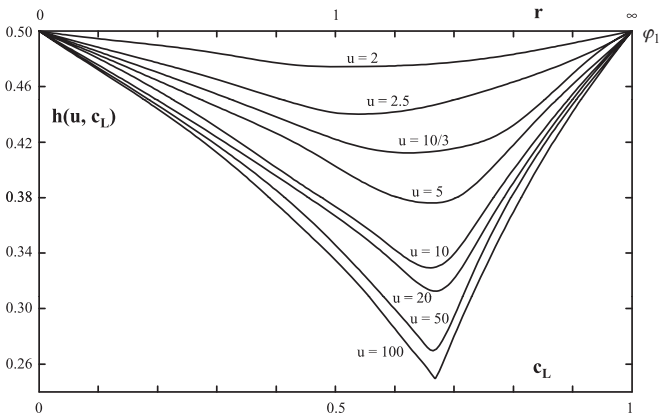

FIG. 1. Void fraction of bimodal mixes (h) of broken particles as a function of size ratio d_L/d_S (u) and volume fraction of large constituent (c_L) according to Furnas [3]. The ratio r [$=c_L/c_S = c_L/(1–c_L)$] is shown as well.

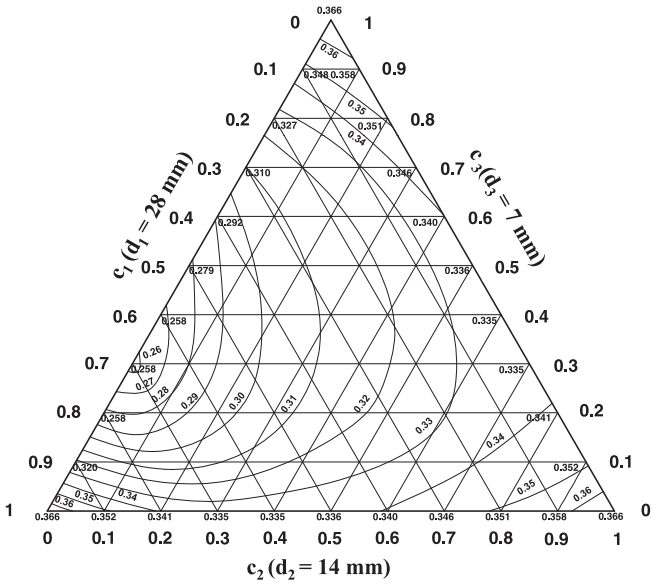

FIG. 2. Ternary void fractions of RCP of sphere mixes as a function of their composition according to Jeschar *et al.* [5]**.**

The compatible unimodal void fractions as measured with the three spheres sizes, as well as the compatible bimodal void fractions measured with d_1/d_2 and d_2/d_3 mixes, indicate that Jeschar *et al.* [5] conducted all their experiments in a reproducible and robust manner.

Using the void fraction of geometric ternary packings ($n = 3$) that can be extracted from Fig. 2, it can now be verified if Eq. (1) is applicable. In Table I the compositions of five geometric packings are included, for which $c_1/c_2 = c_2/c_3 = r$, r being 1/3, 1/2, 1, 2, and 3. The concentrations c_i follow from the cumulative finer fraction [1]:

$$F(d_i) = \frac{d_i^\alpha - d_4^\alpha}{d_1^\alpha - d_4^\alpha}, \tag{2}$$

with $i = 1$, 2, 3, and 4, $d_4 = 3.5$ mm, and α from

$$\alpha = {}^u\log r, \tag{3}$$

in which u is the base of the logarithm. The concentration c_i of each size group follows from $F(d_i)$–$F(d_{i+1})$, and are included in Table I. Equation (2) holds for $\alpha \neq 0$; for $\alpha = 0$ the cumulative finer function reads as follows:

$$F(d) = \frac{{}^u\log d_i - {}^u\log d_4}{{}^u\log d_1 - {}^u\log d_4}. \tag{4}$$

Note that the packing consists of spheres with diameters 28, 14, and 7 mm, but that a size class of 3.5 mm is added (needed) in order to formulate the appropriate cumulative finer functions governed by Eqs. (2) and (4).

The ternary void fraction $j(u, r)$ of each packing as measured by [5] is taken from Fig. 2, and is included in Table I as well. Finally, the ternary void fraction is computed using Eq. (1), whereby the corresponding binary void fraction $h(u, r)$ is taken from Fig. 2 and Table I as well. Comparing the values predicted by Eq. (1) and the measured values (see Table I) reveals that the ternary packing fraction can be predicted accurately from the binary void fraction $h(u, r)$. So, in Ref. [1] it was seen that Eq. (1) holds for saturated discrete packings ($u \to \infty$) and continuous packings ($u \to 1$); now it appears also to hold for geometric packings of discretely sized particles with finite size ratio (here: spheres with $u = 2$).

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TABLE I. Binary (h) and ternary (j) void fractions as measured by Jeschar *et al.* [5] using discretely sized spheres ($d_1 = 28$ mm, $d_2 = 14$ mm and $d_3 = 7$ mm, i.e., $u = 2$). Also the computed void fraction of a ternary geometric packing, using Eq. (1) with $n = 3$ and $\varphi_1 = 0.366$, for various r (being c_i/c_{i+1}, or c_L/c_S in case of the binary mixture) is included.

c_L	c_S	r	$h(u, r)$ measured[a]	α	c_1	c_2	c_3	$j(u, r)$ measured[a]	$j(u, r)$ Eq. (1)
1/4	3/4	1/3	0.348	–1.58	0.08	0.23	0.69	0.334	0.333
1/3	2/3	1/2	0.344	–1	0.14	0.29	0.57	0.323	0.323
1/2	1/2	1	0.335	0	1/3	1/3	1/3	0.313	0.307
2/3	1/3	2	0.335	1	0.57	0.29	0.14	0.308	0.307
3/4	1/4	3	0.338	1.58	0.69	0.23	0.08	0.313	0.312

[a]Reference [5].

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